Midterm Exam Mathematical Physics, Prof. G. Palasantzas

- Total number of points 100
- 10 points for coming to the midterm exam
- Justify your answers for all problems

Problem 1 (20 points)

(a: 10 points) Calculate the limit $\lim_{n\to\infty} a_n$ with $a_n = \left(1 - \frac{y^2}{n}\right)^{9n}$ and y a finite number

(b: 10 points) Is the series $\sum_{n=1}^{\infty} a_n$ convergent ?

Problem 2 (20 points)

Consider a ball that drops from a height h=2 m. Consider that every time it bounces on the ground it will lose 30 % of its energy. Calculate the total vertical distance that the ball travels until it stops on the ground (*take infinite number of bounces until it stops*).

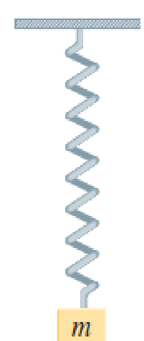
Problem 3 (20 points)

Find the interval (15 points) and the radius (5 points) of convergence of the power series $\sum_{n=1}^{+\infty} \frac{y^n}{1\cdot 3\cdot 5\cdots(2n-1)}$ where y is a finite number.

Problem 4 (30 points)

Forced Oscillations: consider a spring with mass m, spring constant k, and damping constant $c = 0$, and let $\omega = \sqrt{k/m}$. If an external force $F(t) = F_0 \cos \omega t$ is applied (the applied frequency equals the natural frequency), use the method of undetermined coefficients to show that the motion of the mass is given by

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t + \frac{F_0}{2m\omega} t \sin \omega t$$



The general equation of motion of the mass m is: $m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = F(t)$

Problem 1

a)

$$b_n = 9n \ln\left(1 - y^2/n\right)$$

$$\lim_{n\to\infty} b_n = 9 \lim_{n\to\infty} \frac{\frac{d}{dn}\left[\ln\left(1 - y^2/n\right)\right]}{\frac{d}{dn}\left(\frac{1}{n}\right)} \Rightarrow$$

$$\lim_{n\to\infty} b_n = 9 \lim_{n\to\infty} \frac{1/\left(1-y^2/n\right)\left(+y^2/\cancel{n^2}\right)}{\left(-1/\cancel{n^2}\right)} \Rightarrow$$

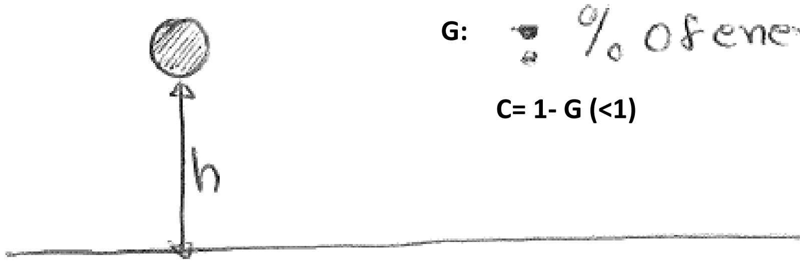
$$\lim_{n\to\infty} b_n = -9y^2 \lim_{n\to\infty} \frac{1}{1-y^2/n} = -9y^2$$

thus we have $\lim_{n\to\infty} a_n = e^{-9y^2} = e^{-(3y)^2}$

b) Because $\lim_{n\to\infty} a_n = e^{-(3y)^2} \neq 0$

the series $\sum_{n=1}^{\infty} a_n$ is divergent

Problem 2



G: % of energy loss

C= 1- G (<1)

$$D = h + \underbrace{2ch}_{\text{1-Jump up-down}} + \underbrace{2c^2h}_{\text{2-Jump up-down}} + \underbrace{2c^3h}_{\text{3-Jump up-down}} + \cdots$$

Thus we have an infinite series

$$D = h + \sum_{n=1}^{\infty} 2c^n h^* = -h + \sum_{n=0}^{\infty} 2c^n h^*$$

$$D = -h + 2h \underbrace{\sum_{n=0}^{\infty} c^n}_{\text{Geometric series}} = -h + 2h \frac{1}{1-c}$$

$$\underline{D = h\frac{1+c}{1-c}}$$

* you can also say $D = h + 2hc\frac{1}{1-c}$

and you have again $\underline{D = h\frac{1+c}{1-c}}$

Problem 3

Simply replace below x with y and you have the solution:

If $a_n = \dfrac{x^n}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}$, then

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}\cdot\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{2n+1} = 0 < 1$. Thus, by the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{x^n}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}$ converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

Problem 4

Forced Vibrations

See in 17.3, the equation of motion : $m\dfrac{d^2x}{dt^2} + c\dfrac{dx}{dt} + kx = F(t)$

And set c=0

The auxiliary equation for the homogenous equation has two imaginary roots $\pm j\omega$ so the solution is: $x_c(t)=c_1\cos\omega t+c_2\sin\omega t$

But the natural frequency of the system equals the

frequency of the external force, so try $x_p(t)=t(A\cos\omega t+B\sin\omega t)$. Then we need $m(2\omega B-\omega^2 At)\cos\omega t-m(2\omega A+\omega^2 Bt)\sin\omega t+kAt\cos\omega t+kBt\sin\omega t=F_0\cos\omega t$ or $2m\omega B=F_0$ and $-2m\omega A=0$ (noting $-m\omega^2 A+kA=0$ and $-m\omega^2 B+kB=0$ since $\omega^2=k/m$). Hence the general solution is

$$x(t)=c_1\cos\omega t+c_2\sin\omega t+\left[F_0 t/(2m\omega)\right]\sin\omega t$$